Mathematical Induction

Part Two

Quick Announcement!

Problem Set Five

- PS5 will be posted soon and is due at the normal Friday 1:00PM time next week.
  - You can use a late day to extend the PS4 deadline to Saturday at 1:00PM if you'd like.
- You know the drill: ask questions on EdStem or office hours if you have them. That's what we're here for!

Outline for Today

- ***"Build Up" versus "Build Down"***
  - An inductive nuance that follows from our general proofwriting principles.
- ***Complete Induction***
  - When one assumption isn't enough!

Recap from Last Time

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

$P(0)$ is true

…and it stays true…

and

$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

then

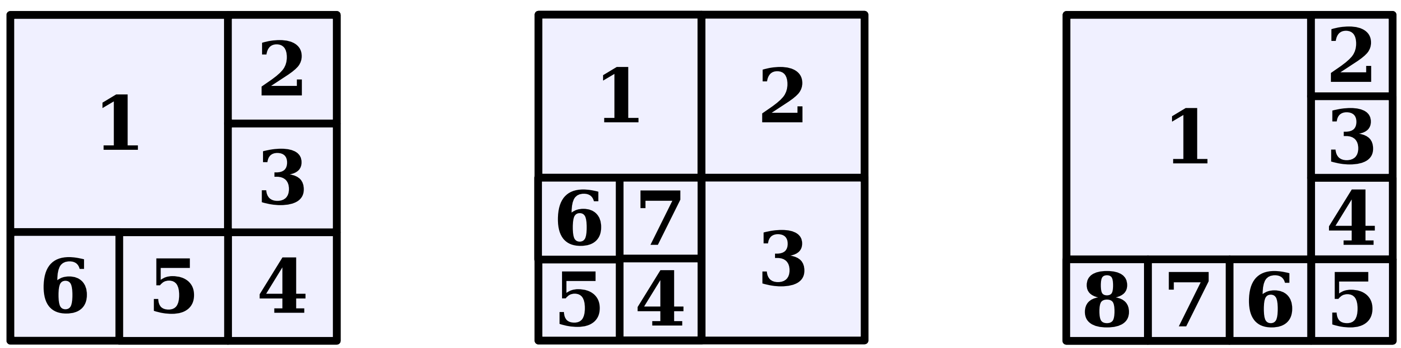
$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

Theorem: For any $n \geq 6$, there is a way to subdivide a square into n smaller squares.

Proof: Let $P(n)$ be the statement "there is a way to subdivide a square into n smaller squares." We will prove by induction that $P(n)$ holds for all $n \geq 6$, from which the theorem follows.

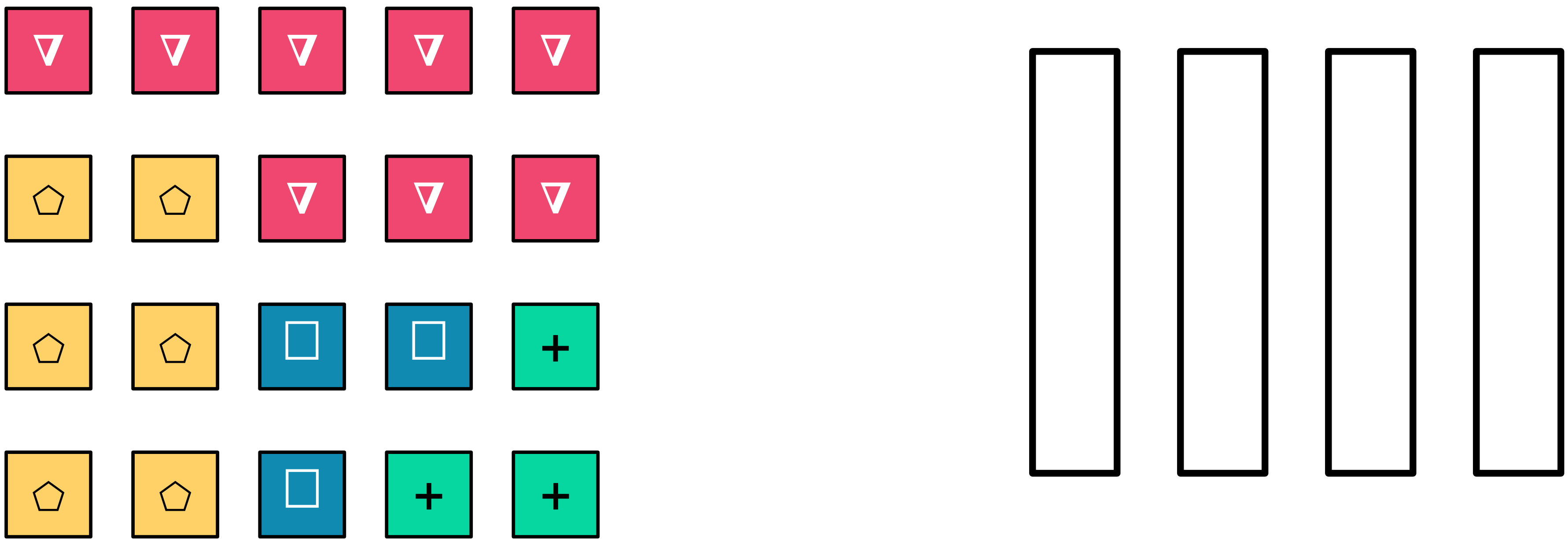
As our base cases, we prove $P(6)$, $P(7)$, and $P(8)$, that a square can be subdivided into 6, 7, and 8 squares. This is shown here:



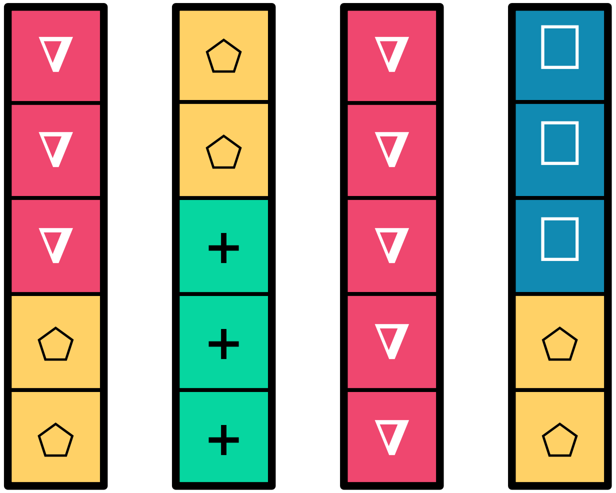
For the inductive step, assume that for some arbitrary $k \geq 6$ that $P(k)$ is true and that there is a way to subdivide a square into k squares. We prove $P(k+3)$, that there is a way to subdivide a square into $k+3$ squares. To see this, start by obtaining (via the inductive hypothesis) a subdivision of a square into k squares. Then, choose any of the squares and split it into four equal squares. This removes one of the k squares and adds four more, so there will be a net total of $k+3$ squares. Thus $P(k+3)$ holds, completing the induction. ■

New Stuff!

The Colored Cubes Problem



Here are 20 cubes of 4 different colors.
Split them into 4 groups of 5 cubes each so that
each group has cubes of at most two different colors.



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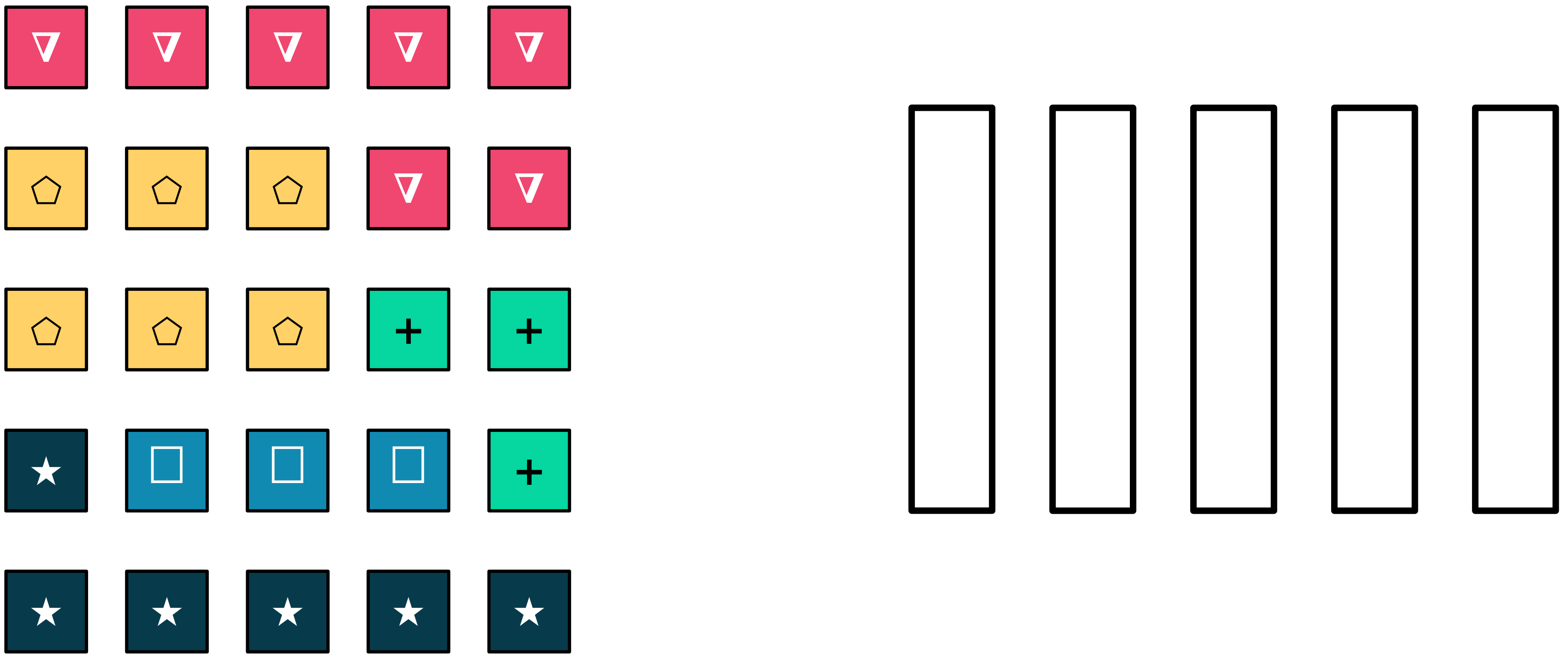
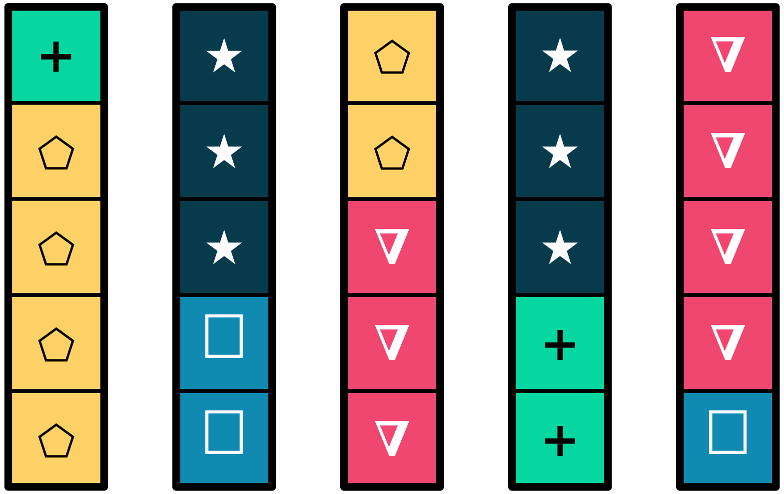


Here are 25 cubes of 5 different colors.
Split them into 5 groups of 5 cubes each so that
each group has cubes of at most two different colors.



Here are 25 cubes of 5 different colors.
Split them into 5 groups of 5 cubes each so that
each group has cubes of at most two different colors.

A ***good split*** of a group of 5n cubes of n colors is a way of splitting them into groups of five each where each group has cubes of at most two colors.

Theorem: For any group of 5n cubes of n colors, there is a good split of those cubes.

$P(n)$ is the statement "for any group of $5n$ cubes of n colors, there exists a good split of those cubes."

$P(0)$

Theorem: For any group of $5n$ cubes of n different colors, there exists a good split of those cubes into groups.

$P(n)$ is the statement "for any group of $5n$ cubes of n colors, there exists a good split of those cubes."

$$\forall k \in \mathbb{N}.\ (P(k) \rightarrow P(k+1))$$

Which of the following best describes the high-level structure of the inductive step of this proof?

A. Begin with a group of $5k$ cubes of k colors.
Find a way to add in five new cubes and one color.

B. Begin with a group of $5k+5$ cubes of $k+1$ colors.
Find a way to remove five cubes and one color.

Answer at ***https://cs103.stanford.edu/pollev***

Theorem: For any group of $5n$ cubes of n different colors, there exists a good split of those cubes into groups.

$P(n)$ is the statement "for any group of $5n$ cubes of n colors, there exists a good split of those cubes."

$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

if

for every group of $5k$ cubes of k colors, there's a good split of those cubes.

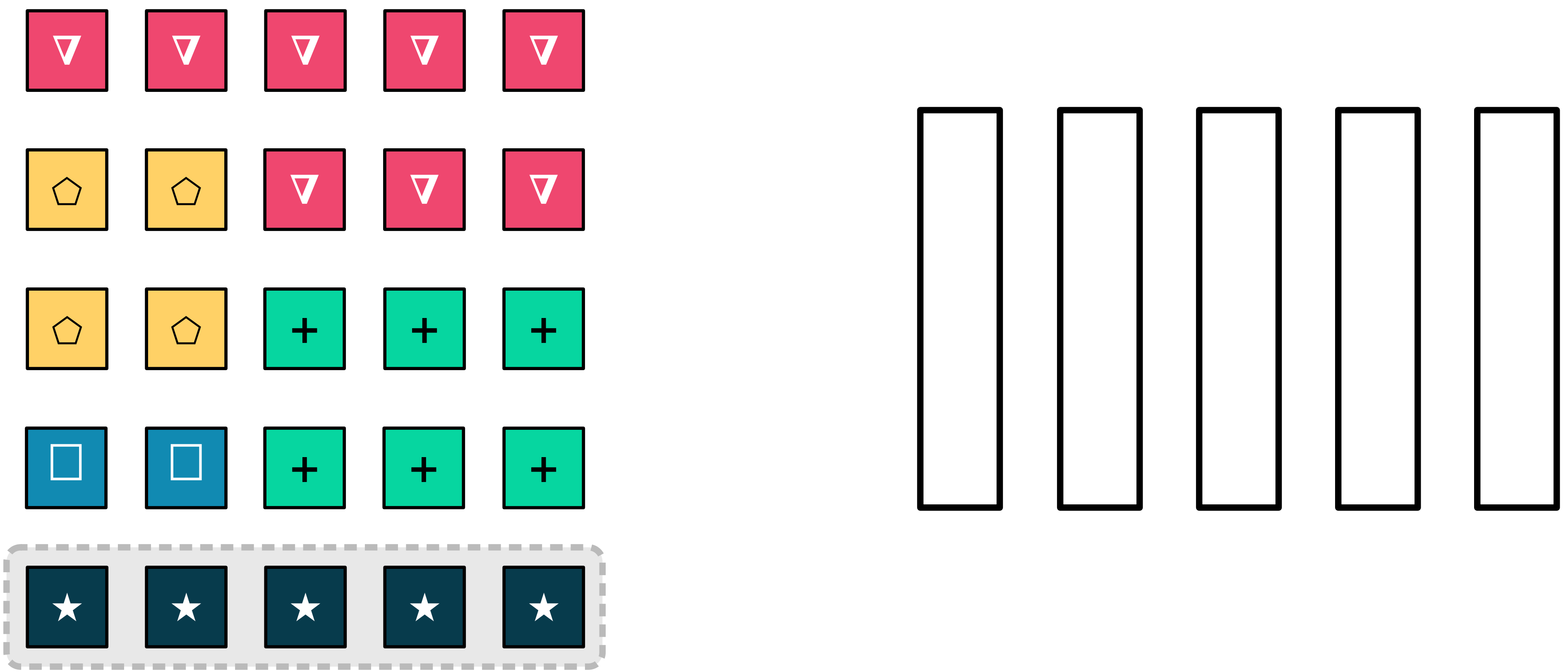
then

for every group of $5k+5$ cubes of $k+1$ colors, there's a good split of those cubes.

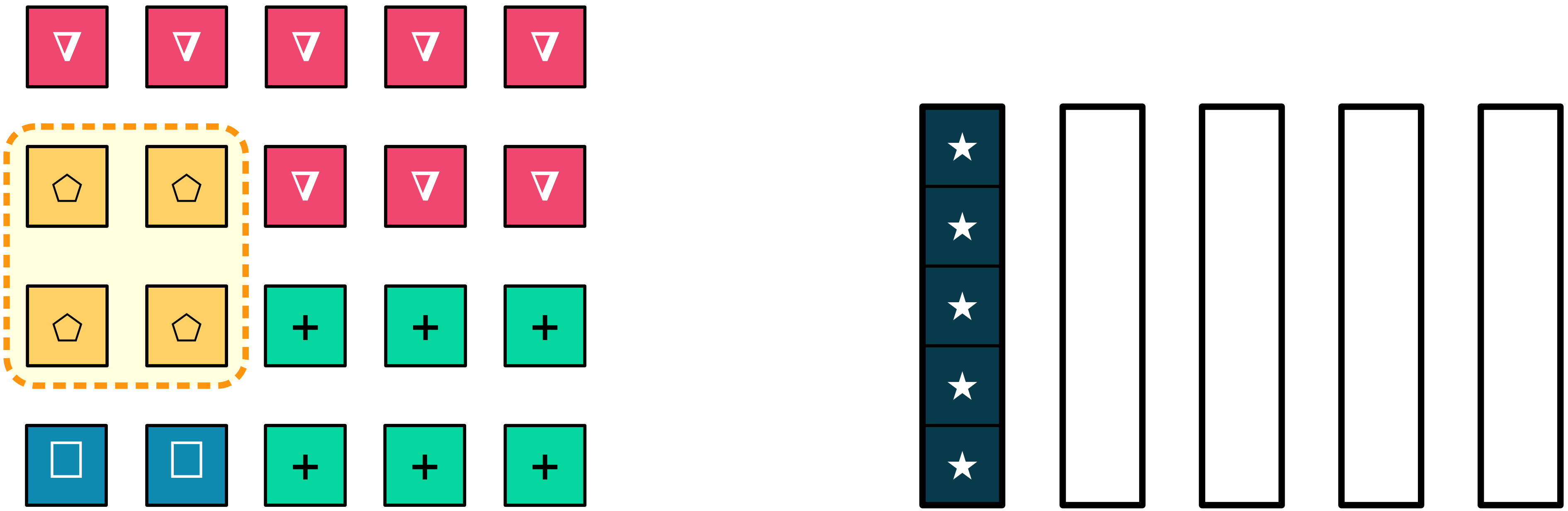
Assume a universal

Prove a universal

Theorem: For any group of $5n$ cubes of n different colors, there exists a good split of those cubes into groups.



Idea: Begin with $5k+5$ cubes and $k+1$ colors.
Find a way to remove five cubes and one color.



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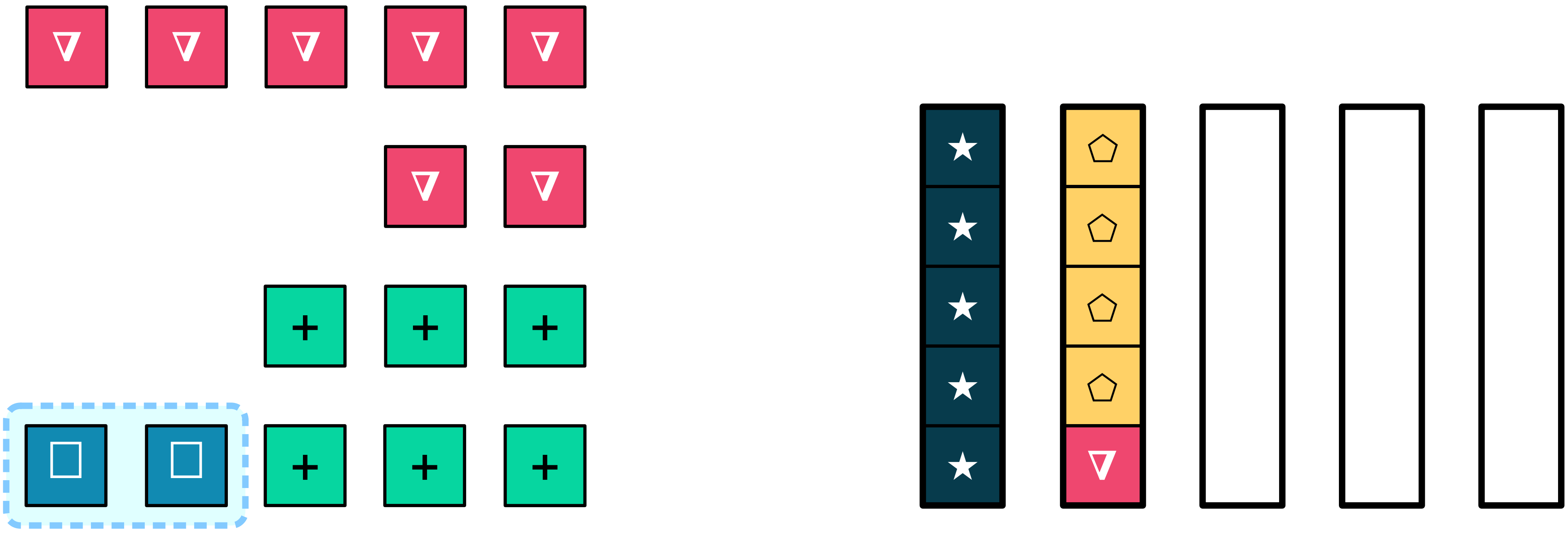

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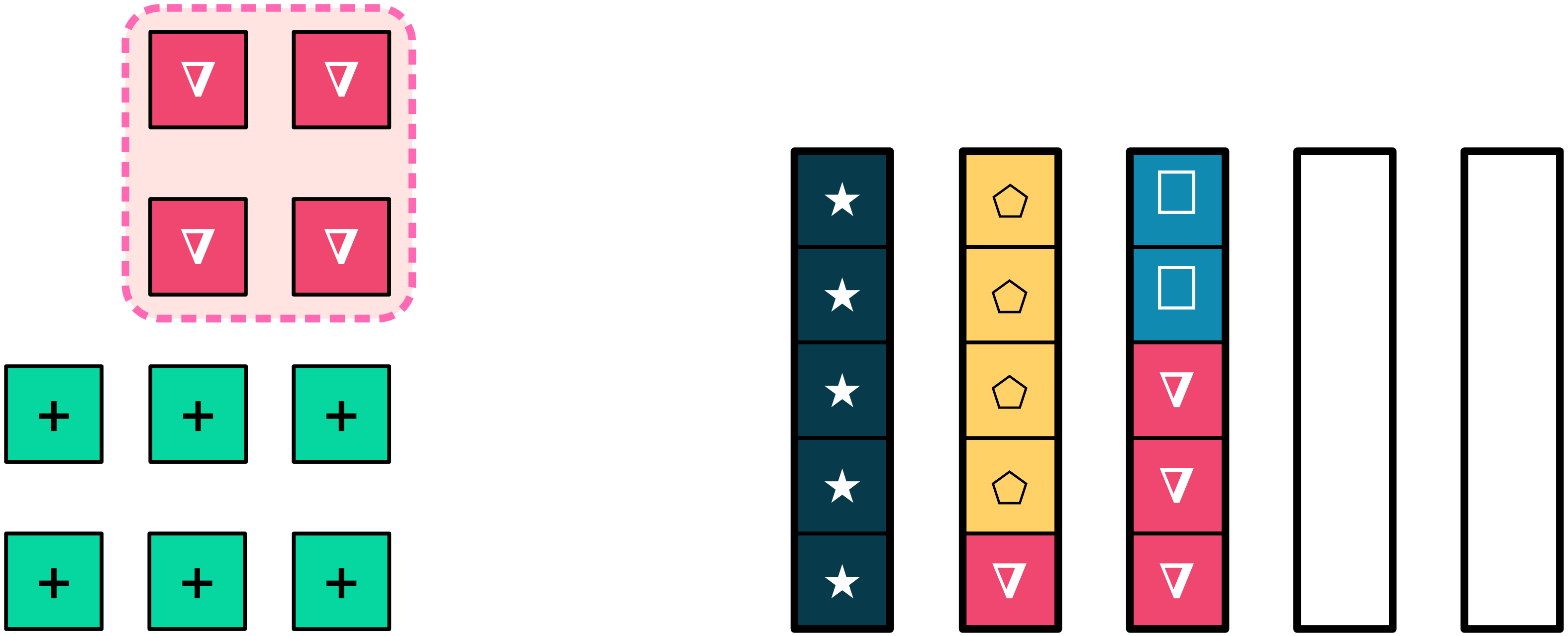

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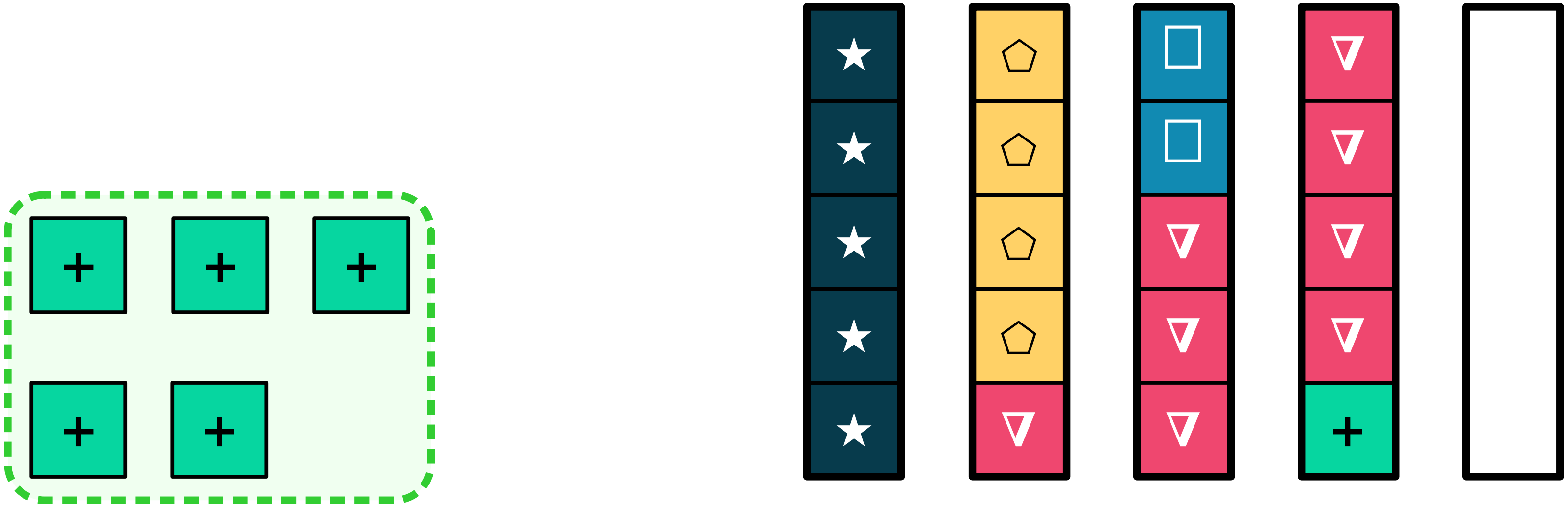
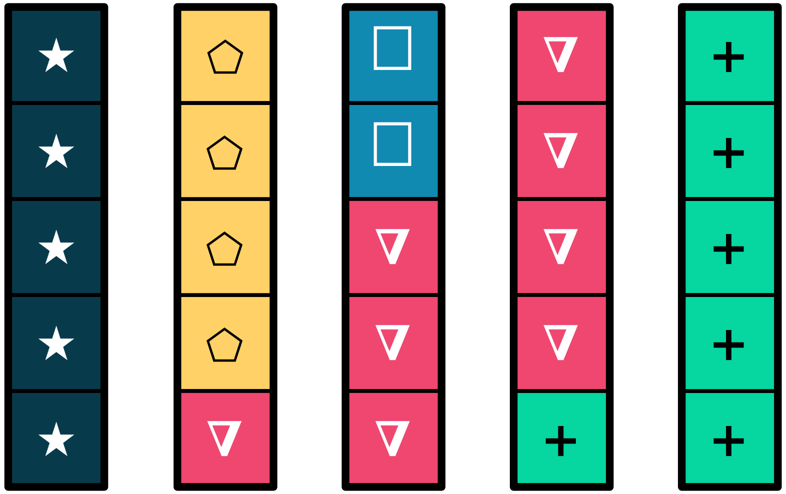


Idea: Begin with $5k+5$ cubes and $k+1$ colors.
Find a way to remove five cubes and one color.

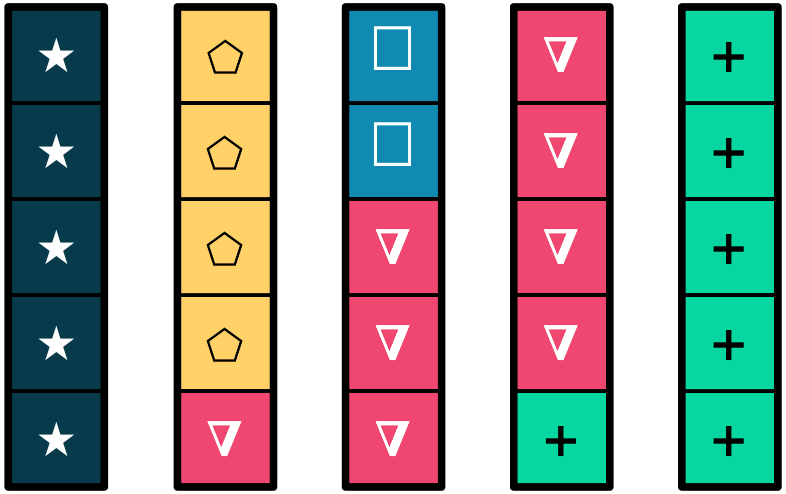


Idea: Begin with $5k+5$ cubes and $k+1$ colors.
Find a way to remove five cubes and one color.

Find a color that appears five or fewer times.

If it's exactly five times, use all cubes of that color in a single group.

Otherwise, use all cubes of that color and "top off" with cubes of another color.



Idea: Begin with $5k+5$ cubes and $k+1$ colors. Find a way to remove five cubes and one color.

Theorem: Every group of $5n$ cubes of n colors has a good split.

Proof: Let $P(n)$ be the statement "for any group of $5n$ cubes of n colors, there exists a good split of those cubes." We will prove that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.

As our base case, we prove $P(0)$, that any group of 0 cubes of 0 colors has a good split. Pick any group of 0 cubes. Placing those cubes into 0 groups satisfies the requirement of a good split, so $P(0)$ holds.

For our inductive step, pick some $k \in \mathbb{N}$ and assume $P(k)$ holds: any group of $5k$ cubes of k colors has a good split. We will prove $P(k+1)$: that any group of $5k+5$ cubes of $k+1$ colors has a good split.

Pick any group of $5k+5$ cubes of $k+1$ colors. By the GPHP, there is a color (call it blue) appearing on $b \leq 5$ cubes.

A nice abbreviation of "generalized pigeonhole principle."

Theorem: Every group of $5n$ cubes of n colors has a good split.

Proof: Let $P(n)$ be the statement "for any group of $5n$ cubes of n colors, there exists a good split of those cubes." We will prove that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.

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Pick any group of $5k+5$ cubes of $k+1$ colors. By the GPHP, there is a color (call it blue) appearing on $b \leq 5$ cubes. We consider two cases:

Case 1: $b = 5$. Place all five blue cubes into their own group.

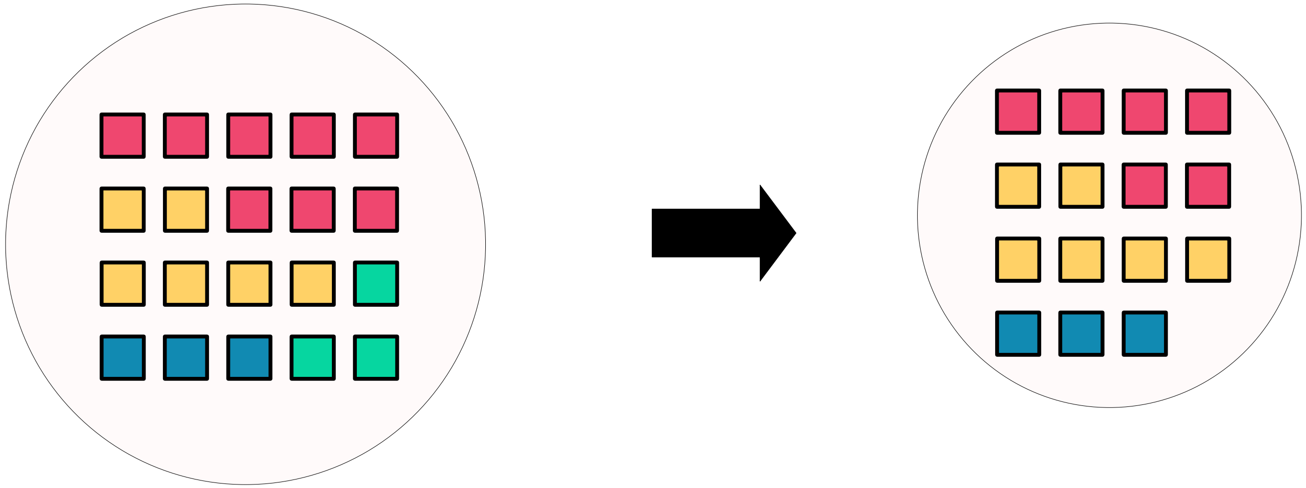
Case 2: $b < 5$. By the GPHP, there is some other color (call it red) appearing on $r \geq 5$ cubes. Place all b blue cubes and $5 - b \leq r$ red cubes into one group.

In each case, we form a group of 5 cubes of at most two different colors and are left with $5k$ cubes of k colors. By our IH, the remaining cubes can be grouped into a good split. That, plus our original group, is a good split of the $5k+5$ cubes. Thus $P(k+1)$ holds, completing the induction. ■

A Neat Application

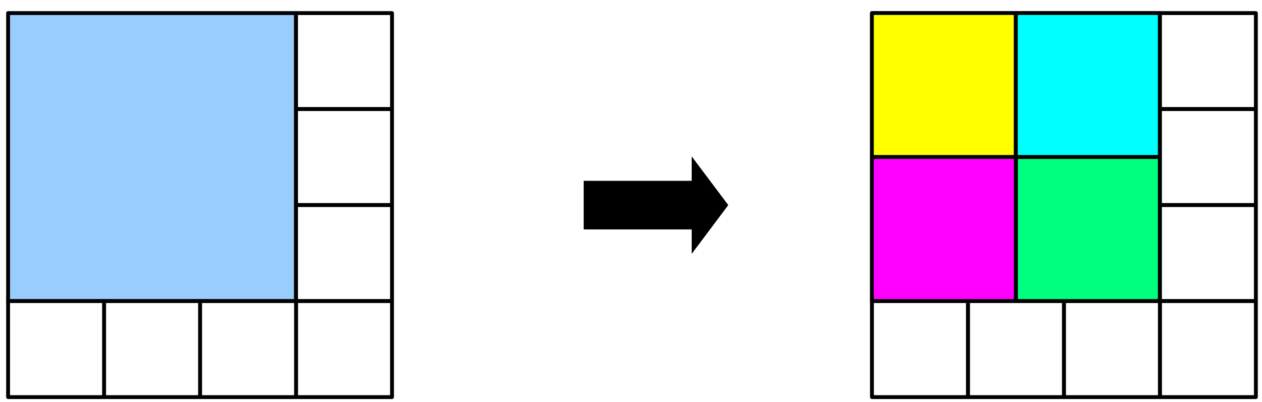
- This result on colored cubes forms the basis for the ***alias method***, a fast algorithm for simulated rolls of a loaded die in software.
- This in turn has applications throughout computer science.
- Want to learn more? Check out ***this blog post***, which shows how to apply this result.

An Observation



Start with more cubes

Get to fewer cubes



Start with fewer squares

Get to more squares

Following the Rules

- When working with square subdivisions, our predicate looked like this:

  $P(n)$ is "***there exists*** a way to subdivide a square into n squares."

- When working with colored cubes, our predicate looked like this:

  $P(n)$ is "***for any*** group of $5n$ cubes of n colors, there is a good split of those cubes."

- With squares, the quantifier is $\exists$. With cubes, the first quantifier is $\forall$.
- This fundamentally changes the "feel" of induction.

Build Up with ∃

- In the case of squares, in our inductive step, we prove

  If

  there exists a subdivision into k squares,

  then

  there exists a subdivision into k+3 squares.

- Assuming the antecedent gives us a concrete subdivision into k squares.
- Proving the consequent means finding some way to subdivide in to k+3 squares.
- The inductive step goal is to "***build up***:" start with a smaller number of squares, and somehow work out what to do to get a larger number of squares.

Build Down with ∀

- In the colored cubes case, in our inductive step, we prove

  If

  for all groups of $5k$ cubes of k colors, there's a good split

  then

  for all groups of $5k+5$ cubes of $k+1$ colors, there's a good split

- Assuming the antecedent means once we find $5k$ cubes and k colors, we can group them into a good split.
- Proving the consequent means picking an arbitrary group of $5k+5$ cubes of $k+1$ colors and looking for a good split.
- The inductive step goal is to "***build down***:" start with a larger set of cubes, then find a way to turn it into a smaller set of cubes.

Some Notes

- Not all predicates $P(n)$ will have the form outlined here.
  - That's okay! Just use the normal rules for assuming and proving things.
  - Think of these as quick shorthands rather than fundamentally new strategies.
- In all cases, assume $P(k)$ and prove $P(k+1)$.
  - All that changes is what you do to assume $P(k)$ and what you do to prove $P(k+1)$.
- When in doubt, consult the assume/prove table.
  - It really does work for all cases!

Complete Induction

Guess what?

It’s time for

Mathematicalesthenics!

What Just Happened?

This is kinda like $P(0)$.

If you are the ***leftmost*** person in your row, stand up right now.

Everyone else: stand up as soon as the person to your left in your row stands up.

This is kinda like $P(k) \to P(k+1)$.

Round Two!

What Just Happened?

This is kinda like P(0).

If you are the ***leftmost*** person in your row, stand up right now.

Everyone else: stand up as soon as ***everyone*** left of you in your row stands up.

What sort of sorcery is this?

Let P be some predicate. The ***principle of complete induction*** states that if

$P(0)$ is true

and

for all $k \in \mathbb{N}$, if $P(0)$, …, and $P(k)$ are true, then $P(k+1)$ is true

then

$\forall n \in \mathbb{N}.\ P(n)$

If it starts true…

…and it stays true…

…then it's always true.

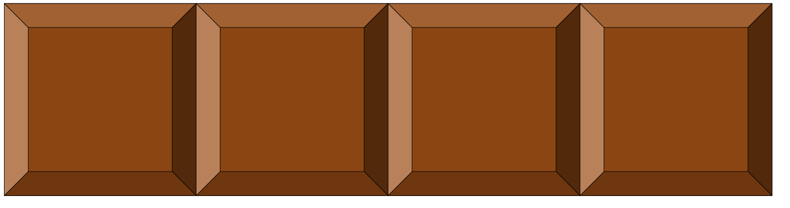
Complete Induction

- You can write proofs using the principle of ***complete*** induction as follows:
  - Define some predicate $P(n)$ to prove by induction on n.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that $\boldsymbol{P(0), P(1), P(2), \ldots,}$ **and** $\boldsymbol{P(k)}$ are all true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

An Example: ***Eating a Chocolate Bar***

Eating a Chocolate Bar

- You have a $1 \times n$ chocolate bar subdivided into 1×1 squares.
- You eat the chocolate bar from left to right by breaking off one or more squares and eating them in one (possibly enormous) bite.
- How many ways can you eat a…
  - 1×1 chocolate bar?
  - 1×2 chocolate bar?
  - 1×3 chocolate bar?
  - 1×4 chocolate bar?



Answer at
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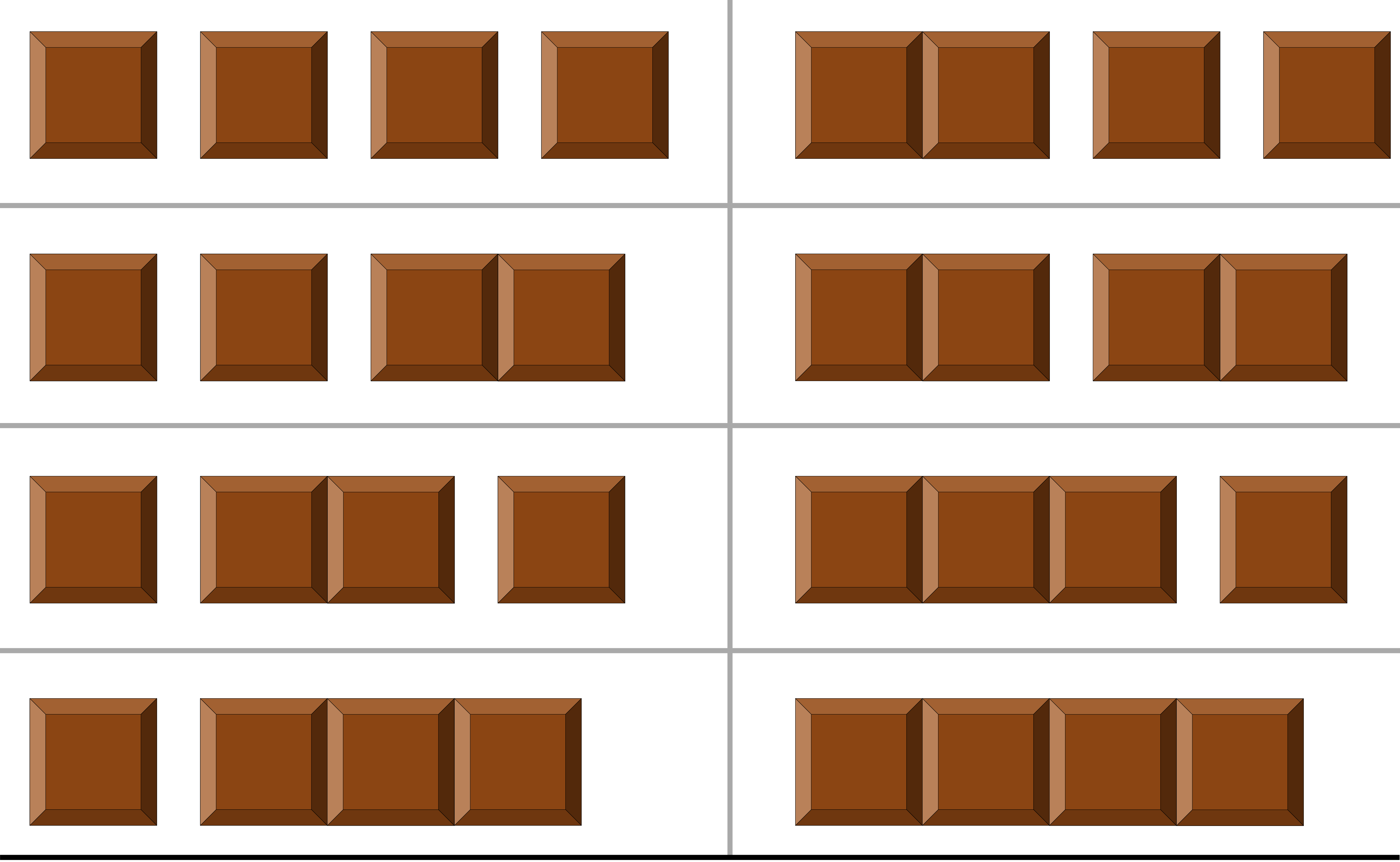
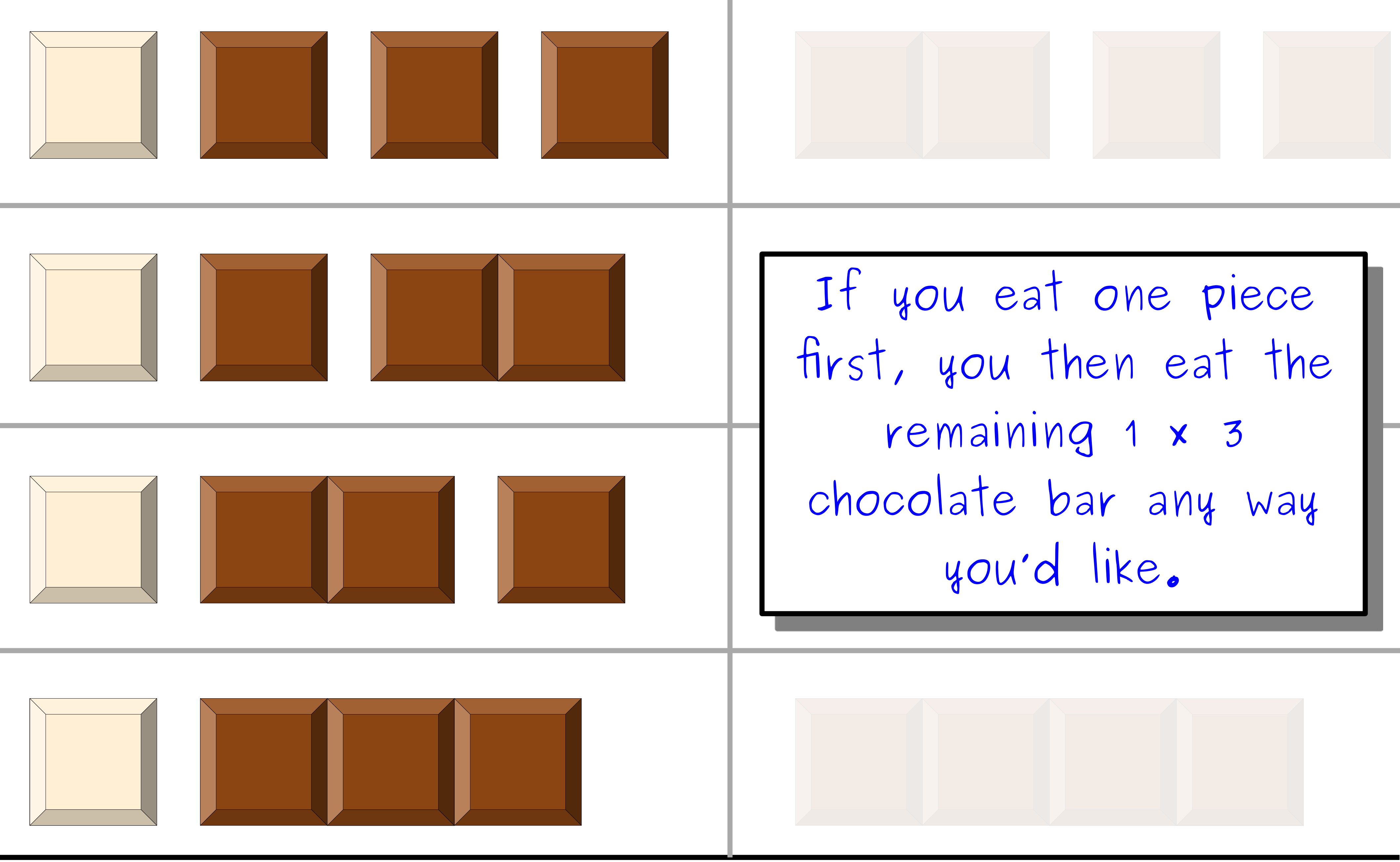


There are eight ways to eat a 1 × 4 chocolate bar.



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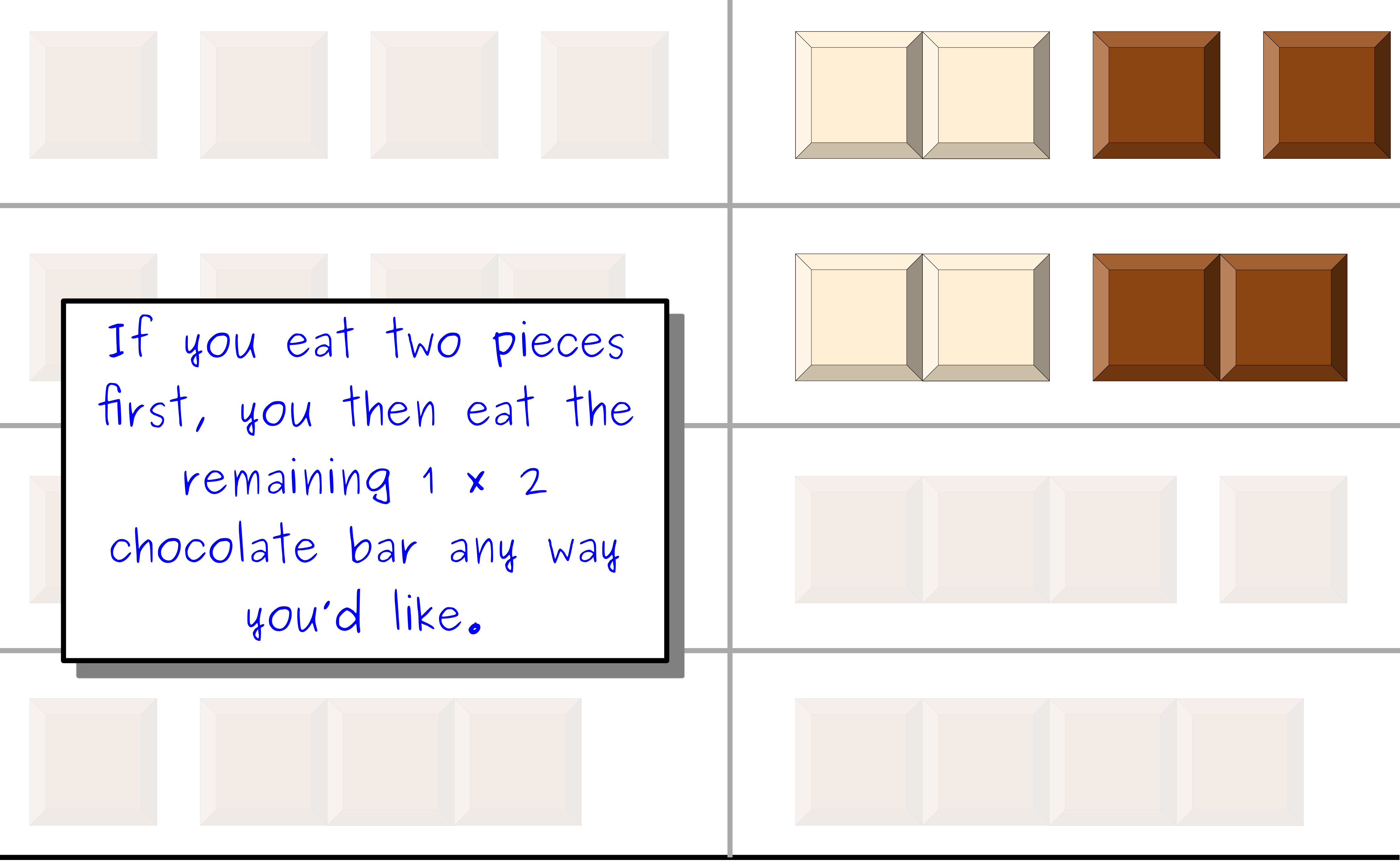

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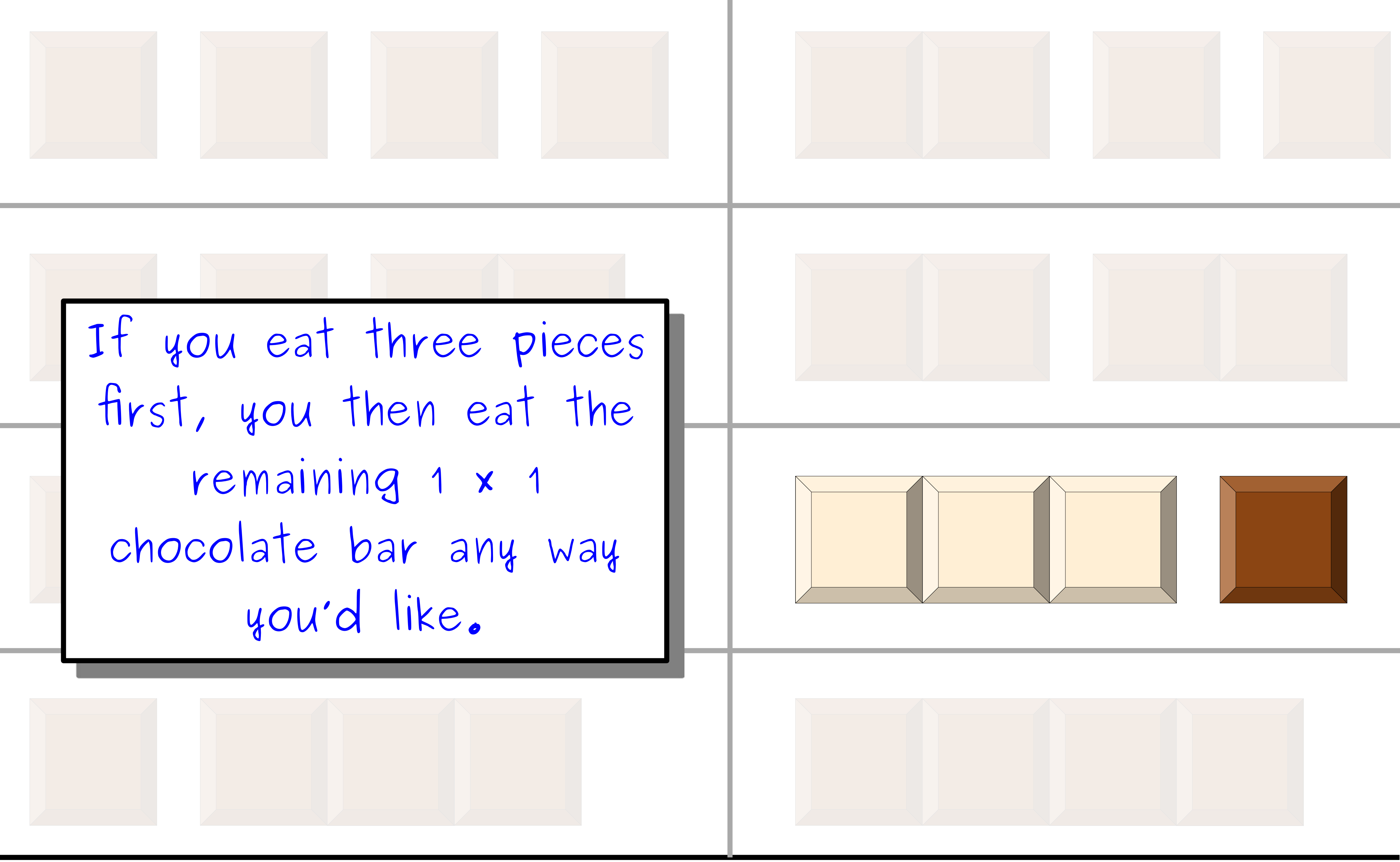

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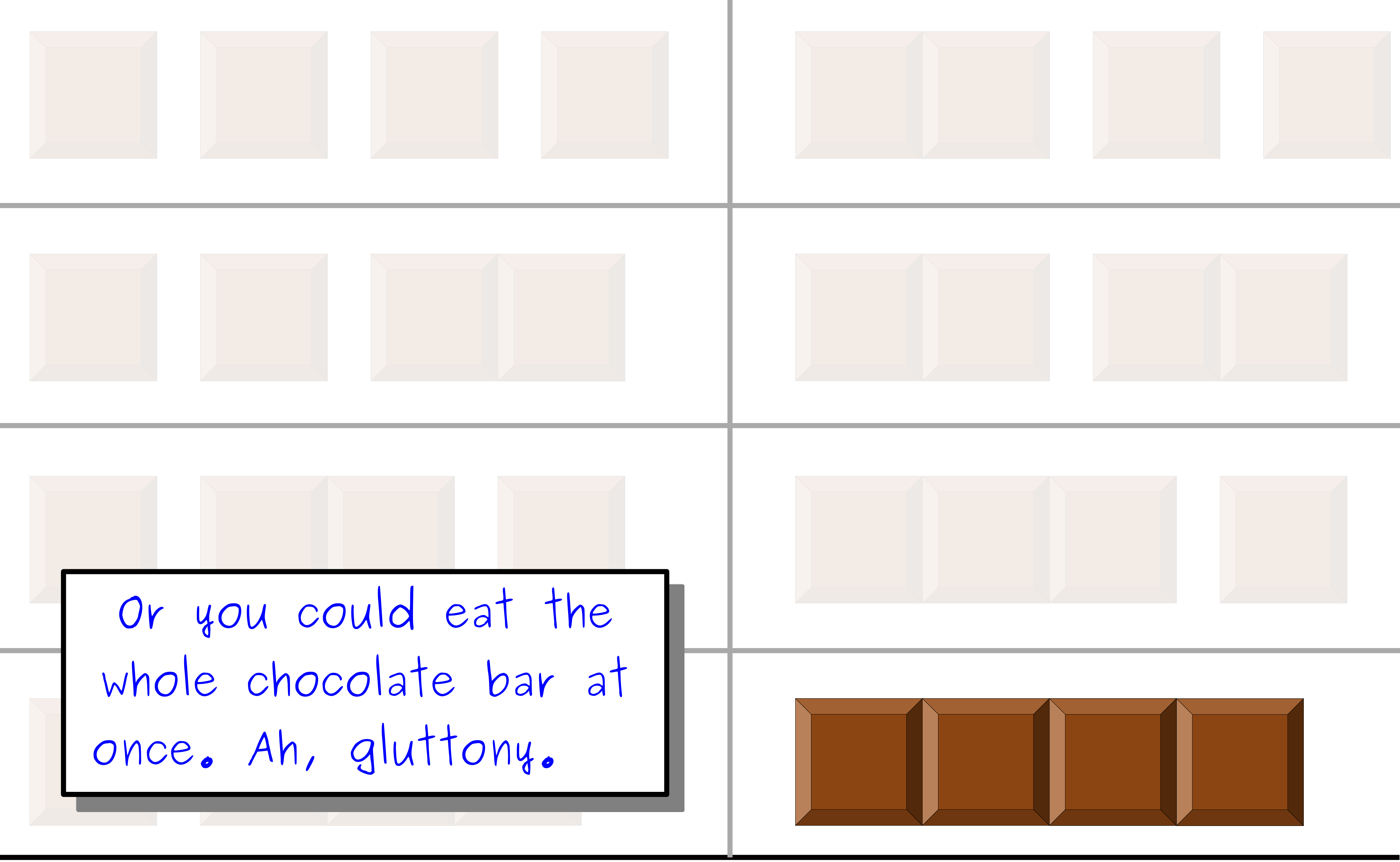


There are eight ways to eat a 1 × 4 chocolate bar.

Eating a Chocolate Bar

- There's…
  - 1 way to eat a 1×1 chocolate bar,
  - 2 ways to eat a 1×2 chocolate bar,
  - 4 ways to eat a 1×3 chocolate bar, and
  - 8 ways to eat a 1×4 chocolate bar.
- ***Our guess:*** There are 2^{n-1} ways to eat a $1 \times n$ chocolate bar for any natural number $n \geq 1$.
- And we think it has something to do with this insight: we eat the bar either by
  - eating the whole thing in one bite, or
  - eating some piece of size k, then eating the remaining $n - k$ pieces however we'd like.
- Let's formalize this!

Theorem: For any natural number $n \geq 1$, the number of ways to eat a $1 \times n$ chocolate bar from left to right is 2^{n-1}.

Proof: Let $P(n)$ be "the number of ways to eat a $1 \times n$ chocolate bar from left to right is 2^{n-1}." We will prove by induction that $P(n)$ holds for all natural numbers $n \geq 1$, from which the theorem follows.

As our base case, we prove $P(1)$, that the number of ways to eat a 1×1 chocolate bar from left to right is $2^{1-1} = 1$. The only option here is to eat the entire chocolate bar at once, so there's just one way to eat it, as needed.

For our inductive step, assume for some arbitrary natural number $k \geq 1$ that $P(1)$, …, and $P(k)$ are true. We need to show $P(k+1)$ is true, that the number of ways to eat a $1 \times (k+1)$ chocolate bar is 2^k.

There are two options for how to eat the bar. First, we can eat the whole chocolate bar in one bite. Second, we could eat a piece of size r for some $1 \leq r \leq k$, leaving a chocolate bar of size $k+1-r$, then eat that chocolate bar from left to right. Since $1 \leq r \leq k$, we know that $1 \leq k+1-r \leq k$, so by our inductive hypothesis there are 2^{k-r} ways to eat the remainder.

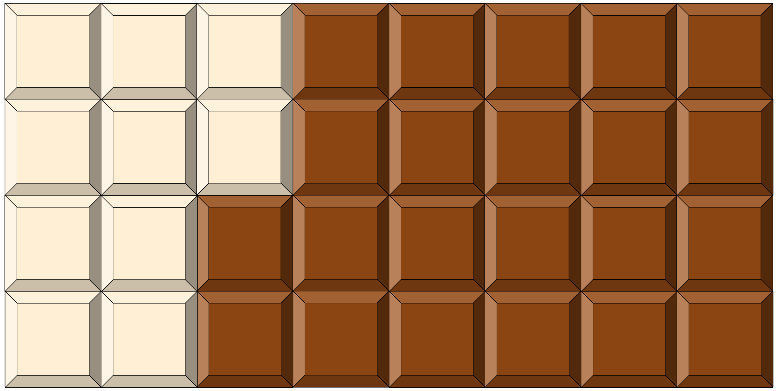
Summing up this first option, plus all choices of r for the second option, we see that the number of ways to eat the chocolate bar is

$$1 + 2^0 + 2^1 + \ldots + 2^{k-1} \quad = \quad 1 + 2^k - 1 \quad = \quad 2^k.$$

Thus $P(k+1)$ holds, completing the induction. ■

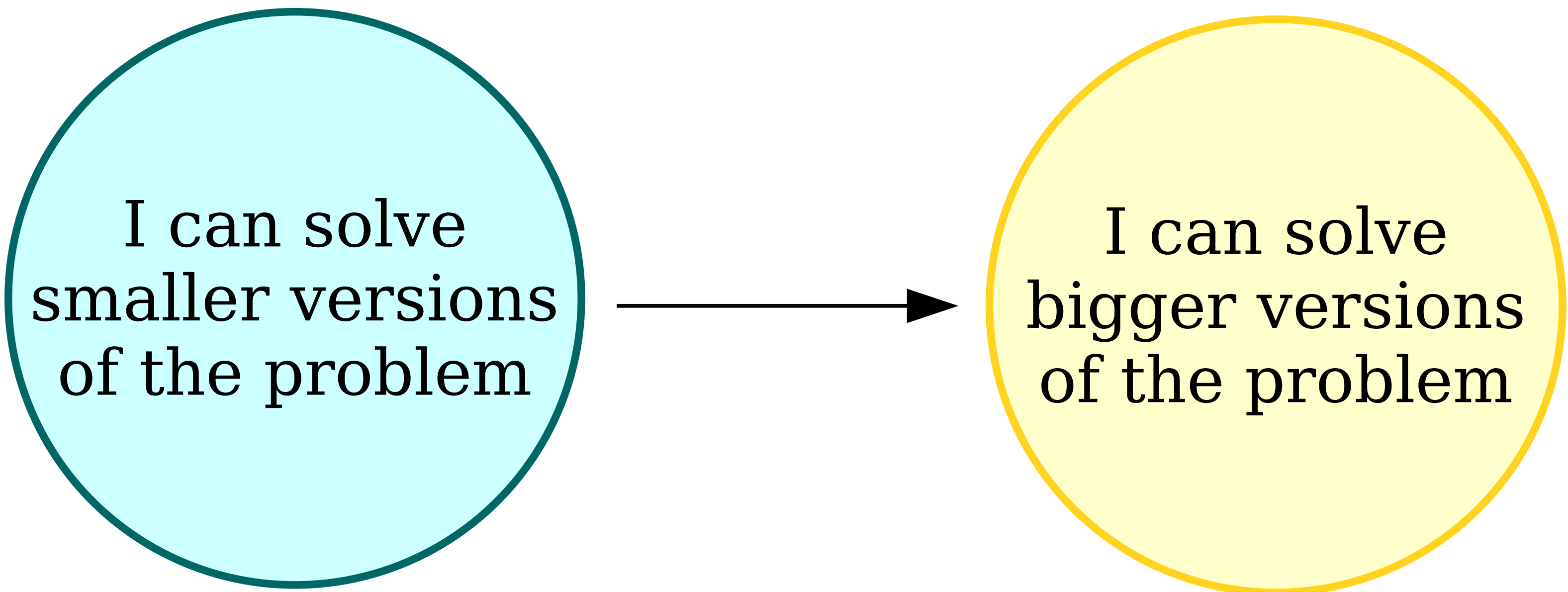
More on Chocolate Bars

- Imagine you have an $m \times n$ chocolate bar. Whenever you eat a square, you have to eat all squares above it and to the left.
- How many ways are there to eat the chocolate bar?

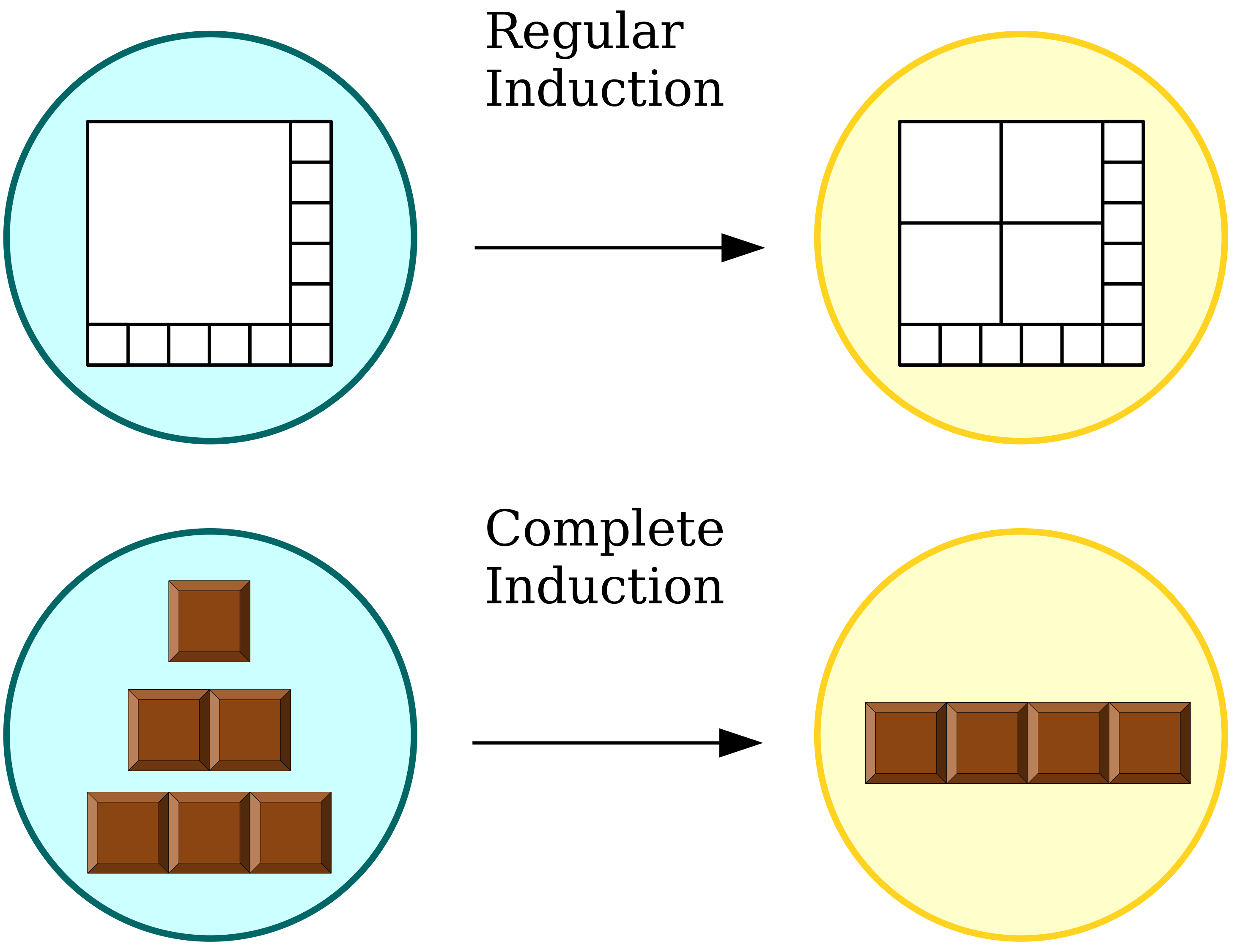


- ***Open Problem:*** Find a non-recursive exact formula for this number, or give an approximation whose error drops to zero as m and n tend toward infinity.

Induction vs. Complete Induction



Induction vs. Complete Induction



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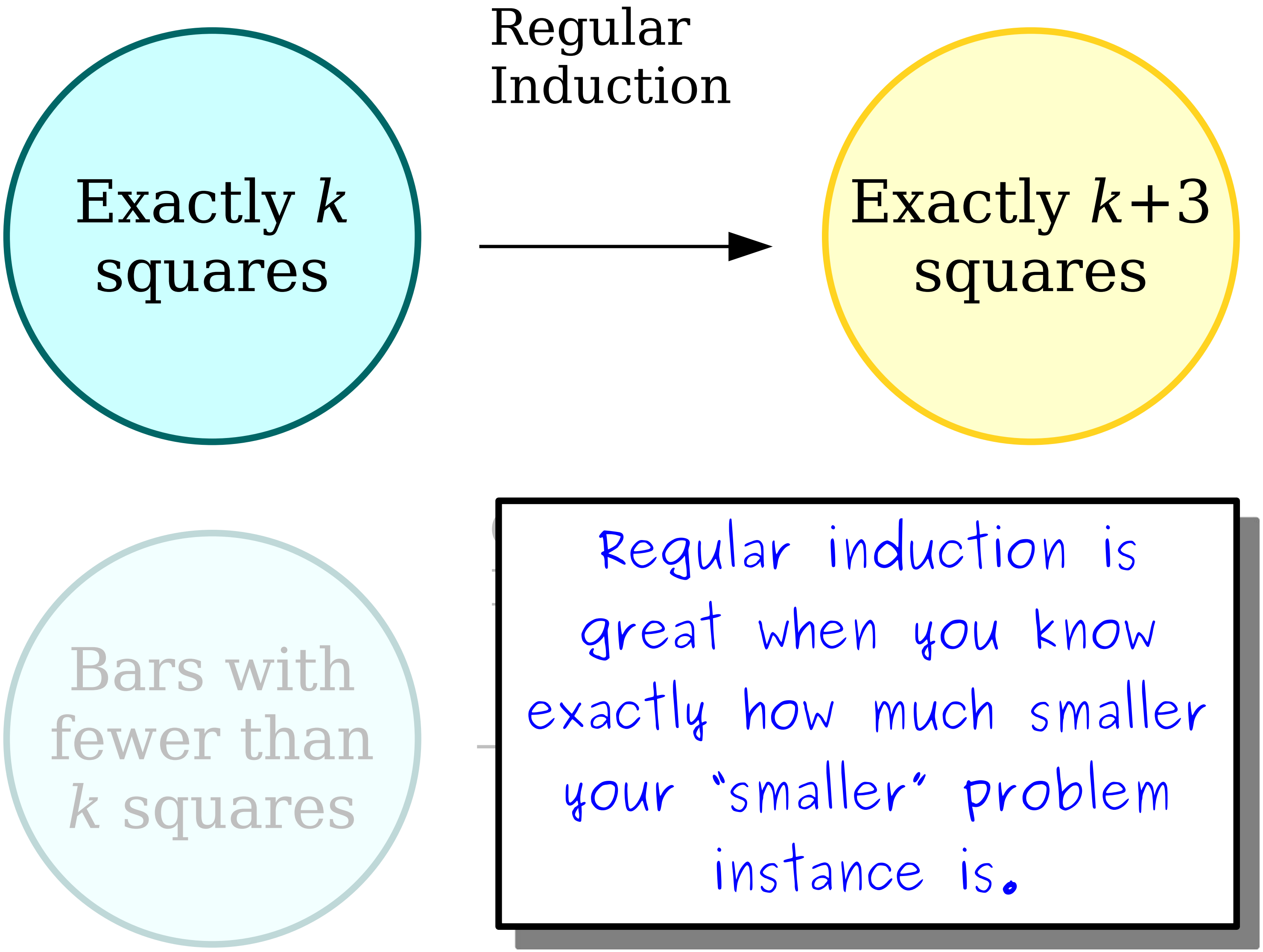
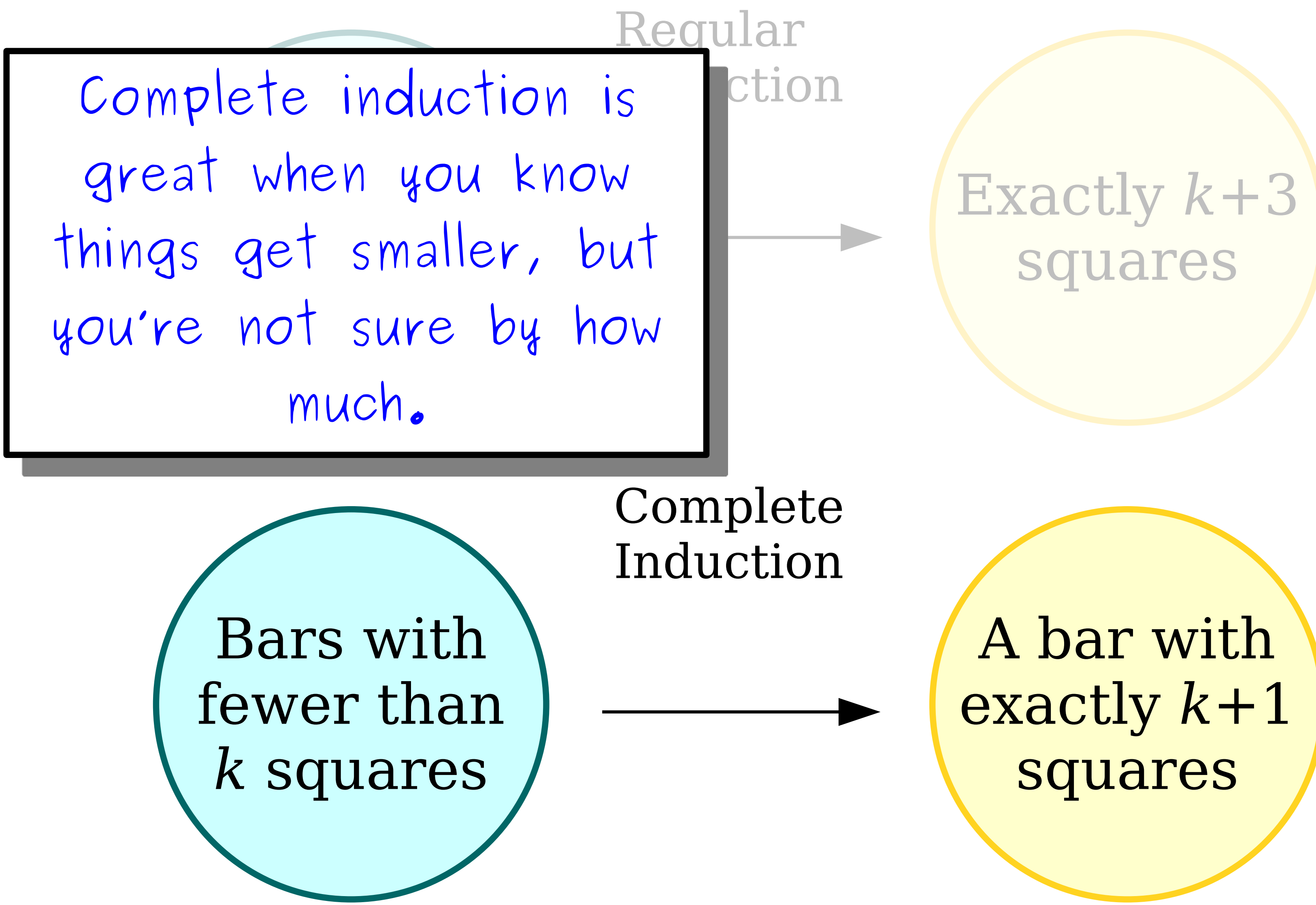


Induction vs. Complete Induction



An Important Milestone

Recap: ***Discrete Mathematics***

- The past five weeks have focused exclusively on discrete mathematics:

Induction	Functions
Graphs	The Pigeonhole Principle
Formal Proofs	Mathematical Logic
Set Theory	

- These are building blocks we will use throughout the rest of the quarter.
- These are building blocks you will use throughout the rest of your CS career.

Next Up: ***Computability Theory***

- It's time to switch gears and address the limits of what can be computed.
- We'll explore these questions:
  - How do we model computation itself?
  - What exactly is a computing device?
  - What problems can be solved by computers?
  - What problems *can't* be solved by computers?
- ***Get ready to explore the boundaries of what computers could ever be made to do.***

Next Time

- ***Formal Language Theory***
  - How are we going to formally model computation?
- ***Finite Automata***
  - A simple but powerful computing device made entirely of math!
- ***DFAs***
  - A fundamental building block in computing.